Vector Bundles
vs.
Locally Free Sheaves

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Abstract

Algebraic geometers usually switch effortlessly between the notion of a **vector bundle** and a **locally free sheaf**. I will define both terms, assuming a good knowledge of [Har, Chapter II] and explain why they are virtually the same. This is essentially [Har, Exercise II.5.18], but my Definition 1.1 of a vector bundle might be easier to recognize by someone with less background in algebraic geometry. Also, the correspondence established does not mention the sheaf of sections but uses the natural scheme structure of GL_n to establish the correspondence instead.

1 Setting up shop

All schemes will be over a commutative ring $\Bbbk$. You may very well think of $\Bbbk$ as a field. Throughout, let X be such a scheme. Fiber products will be with respect to $\mathrm{Spec}(\Bbbk)$. We define $\mathbb{A}^n := \mathbb{A}^n_{\Bbbk} = \mathrm{Spec}(\Bbbk[X_1, \ldots, X_n])$ and $\mathbb{A}^n_X := X \times \mathbb{A}^n_{\Bbbk}$. We denote by $\mathrm{pr}_X : \mathbb{A}^n_X \to X$ the projection to X and $\mathrm{pr}_{\mathbb{A}}$ the projection to $\mathbb{A}^n$. While both notations are slightly abusive, it should always be very clear from the context what is ment.

Let $n \in \mathbb{N}$ be a natural number and $G := \mathrm{GL}_n(\Bbbk)$ the group of invertible matrices over $\Bbbk$, interpreted as a group scheme. It comes with the action morphism $\alpha : \mathbb{A}^n_G \to \mathbb{A}^n$, the multiplication (composition) morphism $\mu : G \times G \to G$ and the inversion morphism $\iota : G \to G$.

Definition 1.1 (Vector Bundle). A scheme E with a structural morphism $h : E \to X$ is called a **vector bundle of dimension** n if there exists an open covering $X = \bigcup_{U \in \mathcal{U}} U$ such that

(1) For each $U \in \mathcal{U}$, there exists an isomorphism $r_U : h^{-1}(U) \to \mathbb{A}^n_U$ with $\mathrm{pr}_U \circ r_U = h|_{h^{-1}(U)}$. In diagram form,

$$\begin{tikzcd} \mathbb{A}^n_U \arrow[dr, "\mathrm{pr}_U"'] & h^{-1}(U) \arrow[l, "r_U"'] \arrow[r, hook] \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] & E \arrow[d, "h"] \arrow[dl, phantom, "\circlearrowleft"] \\ & U \arrow[r, hook] & X \end{tikzcd} \tag{1.1}$$

(2) For $U, V \in \mathcal{U}$, the map $r_{UV} : \mathbb{A}^n_{U\cap V} \to \mathbb{A}^n_{V \cap U}$, defined by[1]

$$\begin{tikzcd} \mathbb{A}^n_{U\cap V} \arrow[d, "r_{UV}"'] & \\ \mathbb{A}^n_{U\cap V} & U \cap V \arrow[ul, "r_V"'] \arrow[l, "r_U"] \arrow[ull, phantom, "\circlearrowleft"] \end{tikzcd} \tag{1.2}$$

is linear on the fibers. This means that there is a morphism of schemes $\gamma_{UV} : U \cap V \to G$ making the following diagram commutative[2]:

$$\begin{tikzcd} (U\cap V) \times \mathbb{A}^n \arrow[rr, "\gamma_{UV} \times \mathrm{id}"] \arrow[dr, dotted, "r_{UV}"] \arrow[ddr, bend right, "\mathrm{pr}_{U\cap V}"'] & & G \times \mathbb{A}^n \arrow[d, "\alpha"] \\ & (U \cap V) \times \mathbb{A}^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & \mathbb{A}^n \arrow[d] \\ & U \cap V \arrow[r] & \mathrm{Spec}(\mathbb{k}). \end{tikzcd} \tag{1.3}$$

[1]In other words, $r_{UV} := r_U \circ r_V^{-1}$.
[2]Intuitively, $r_{UV}(P, a) = (P, \gamma_{UV}(P).a)$ for $P \in U \cap V$ and $a \in \mathbb{A}^n$.

Fact 1.2. *For $U, V, W \in \mathcal{U}$, we have $r_{WU} = r_{WV} \circ r_{VU}$ on $U \cap V \cap W$.*

Proof. $r_{WV} \circ r_{VU} = r_W \circ r_V^{-1} \circ r_V \circ r_U^{-1} = r_W \circ r_U^{-1} = r_{WU}$. ■

Definition 1.3. Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules. Then, $\mathcal{E}$ is said to be **locally free of rank** n if for each $P \in X$, the stalk $\mathcal{E}_P$ is a free $\mathcal{O}_{X,P}$-module of rank n. ●

2 Automorphisms of free $\mathcal{O}_X$-modules

In this section, we will establish a one-to-one correspondence

$$\mathrm{Aut}_{\mathcal{O}_X}(\mathcal{O}_X^n) \cong \mathcal{S}c\mathcal{h}(X, G).$$

between the automorphisms of $\mathcal{O}_X^n$ (the free $\mathcal{O}_X$-module of rank n) and the morphisms $X \to G$ of schemes.

Lemma 2.1. *Let X be a scheme. Then, $\mathcal{O}_X(X)$ is in bijection with the morphisms $X \to \mathbb{A}^1$ of schemes.*

Proof. Let $A = \Bbbk[T]$ be the univariate polynomial ring over $\Bbbk$. Note that $\mathrm{Spec}(A) = \mathbb{A}^1$. By [Har, Exercise II.2.4], the morphisms $X \to \mathrm{Spec}(A)$ are in bijection with the ring homomorphisms $A \to \mathcal{O}_X(X)$. Hence, the morphisms $X \to \mathbb{A}^1$ over $\Bbbk$ are in bijection with the $\Bbbk$-algebra homomorphisms $\Bbbk[T] \to \mathcal{O}_X(X)$, but these are in bijection with $\mathcal{O}_X(X)$ itself, by the universal property of the polynomial ring. ■

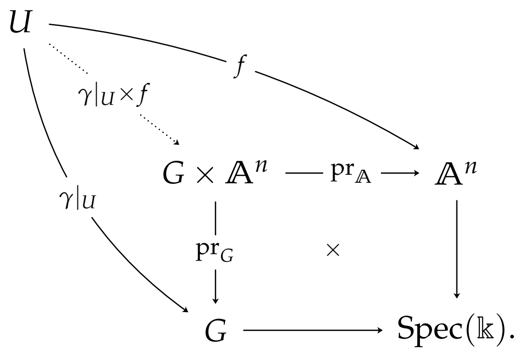
First, any morphism $\gamma : X \to G$ of schemes yields an isomorphism

$$\phi_\gamma : \mathcal{O}_X^n \xrightarrow{\sim} \mathcal{O}_X^n \tag{2.1}$$

which is given on any open $U \subseteq X$ by the rule

$$\begin{aligned} \mathcal{O}_X(U)^n &\xrightarrow{\sim} \mathcal{O}_X(U)^n \\ f &\longmapsto \alpha \circ (\gamma|_U \times f). \end{aligned}$$

Here, we use Lemma 2.1 to identify $f \in \mathcal{O}_X(U)^n$ with a morphism of schemes $U \to \mathbb{A}^n$, denoted also by the symbol f, and $\gamma|_U \times f$ is the unique morphism that makes the following diagram commute:



The fact that ϕ_γ is an isomorphism follows because an inverse is given by $\phi_{\iota\circ\gamma}$. Indeed, denoting by $\mathbf{1} : X \to G$ the constant function that maps closed points to the neutral element $\mathbf{1} \in G$,

$$\begin{aligned} \phi_{\iota\circ\gamma}(\phi_\gamma(f)) &= \alpha \circ ((\iota \circ \gamma) \times \alpha \circ (\gamma \times f)) \\ &= \alpha \circ (\mu \circ (\iota \circ \gamma \times \gamma) \times f) = \alpha \circ (\mu \circ (\iota \times \mathrm{id}) \circ \gamma \times f) \\ &= \alpha \circ (\mathbf{1} \circ \gamma \times f) = \alpha \circ (\mathbf{1} \times f) = f. \end{aligned}$$

On the other hand, we claim that any isomorphism $\phi : \mathcal{O}_X^n \xrightarrow{\sim} \mathcal{O}_X^n$ is of this form: For every open affine subset $U = \mathrm{Spec}(A)$ of X, we get an automorphism $\phi_U \in \mathrm{Aut}_A(A^n)$. Thus, ϕ_U can be represented by a matrix $\gamma_U \in \mathrm{GL}_n(A)$, which is essentially a matrix of morphisms $U \to \mathbb{A}^1$. We can instead understand it as a matrix-valued morphism $\gamma_U : U \to G$. Glueing the γ_U yields a $\gamma : X \to G$ which satisfies $\phi_\gamma = \phi$.

3 The correspondence

Given a vector bundle $h : E \to X$ with notation as in Definition 1.1, we have isomorphisms (see (2.1))

$$\phi_{UV} = \phi_{\gamma_{UV}} : \mathcal{O}_U^n|_{U\cap V} \xrightarrow{\sim} \mathcal{O}_V^n|_{V\cap U}$$

It is clear that $\phi_{UV} = \phi_{VU}^{-1}$ and Fact 1.2 yields the second condition for glueing the sheaves $\mathcal{O}_U^n$ along the ϕ_{UV}. See [Har, Exercise 1.22]. We call the resulting sheaf $\mathcal{E}$. It is clear that $\mathcal{E}$ is locally free of rank n.

Given a locally free sheaf $\mathcal{E}$ on X, we choose an open, affine covering $\mathcal{U}$ of X such that for all $U \in \mathcal{U}$, we have isomorphisms $\phi_U : \mathcal{E}|_U \xrightarrow{\sim} \mathcal{O}_U^n$. For $U, V \in \mathcal{U}$, we define $\phi_{UV} := \phi_U|_{U\cap V} \circ \phi_V|_{V\cap U}^{-1}$ mapping

$$\phi_{UV} : \mathcal{O}_V^n|_{V\cap U} \xrightarrow{\sim} \mathcal{O}_U^n|_{U\cap V}.$$

By section 2, we know that there exist regular morphisms $\gamma_{UV} : X \to G$ with $\phi_{UV}(f) = \alpha \circ (\gamma_{UV} \times f)$. From γ_{UV} we get $r_{UV} : \mathbb{A}^n_{V\cap U} \xrightarrow{\sim} \mathbb{A}^n_{U\cap V}$, simply *defined* via (1.3). These morphisms can be used to glue the schemes $\mathbb{A}^n_U$ to a scheme E. Simultaneously, we glue the $\mathbb{A}^n_U \to U \hookrightarrow X$ to a structural morphism $h : E \to X$. The glueing procedure[3] yields morphisms $t_U : \mathbb{A}^n_U \to E$ which are isomorphisms onto an open subset of E with $t_V = t_U \circ r_{UV}$ on $U \cap V$. All the while, the glued morphism h satisfies

$$\begin{array}{ccc} \mathbb{A}^n_U & \xrightarrow{t_U} & E \\ \downarrow{\scriptstyle \mathrm{pr}_U} & \circlearrowleft & \downarrow{\scriptstyle h} \\ U & \longrightarrow & X. \end{array}$$

Hence, $\mathrm{im}(t_U) = h^{-1}(U)$. We denote by $r_U : h^{-1}(U) \to \mathbb{A}^n_U$ the inverse of t_U and hence, $r_U = r_{UV} \circ r_V$ as required for (1.1) and (1.2).

It should be quite clear that these two operations are inverse to each other from the results of section 2.

References

[Har] Robin Hartshorne. *Algebraic Geometry*. Springer, New York, 2006.

[3] C.f. [Har, Exercise II.2.12]